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An Exact Discount Factor Restriction for Period-Three Cycles in Dynamic Optimization Models*

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This paper explores the precise extent of discounting needed to generate period-three cycles in a standard aggregative dynamic optimization framework. It is shown that there is a "universal constant", $M \equiv [(\sqrt{5}-1)/2]^2 \approx 0.3819$, such that (i) if an optimal program of any dynamic optimization model exhibits a period-three cycle, then the discount factor is less than M; and (ii) if the discount factor is smaller than M, then it is possible to construct a transition possibility set and a utility function such that the resulting dynamic optimization model exhibits a period-three cycle. *Journal of Economic Literature* Classification Numbers: C61, E32, O41.

1. INTRODUCTION

Consider a standard aggregative dynamic optimization framework (Ω, u, δ), where Ω is the transition possibility (technology) set, u is a (reduced form) utility function defined on this set, and $0 < \delta < 1$ a discount factor. Can an optimal program in this framework exhibit a period-three cycle

I will show, in this paper, that a complete answer to this question can be given as follows. There is a "universal constant"[1] defined by

$$M \equiv [(\sqrt{5}-1)/2]^2 \approx 0.3819 \tag{1.1}$$

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[1] The constant, M, is closely related to the well-known "golden ratio," Φ. We recall, briefly, how the golden ratio is defined. If we divide a line into two parts (A and B) such that the ratio of A to B is the same as the ratio of the line itself to A, then the ratio of A to B is called the *golden ratio*. The relevant equation to solve is: $(A/B) = (A+B)/A$. Denoting (A/B) by x, we get the equation: $x = 1 + (1/x)$, which is a quadratic: $x^2 - x - 1 = 0$. The positive solution (denoted by Φ) is given by $\Phi = [1+\sqrt{5}]/2 \approx 1.618$. [The number, Φ, figures prominently in Fibonacci series: 1, 1, 2, 3, 5, 8, 13, 21, 34, ..., where each entry is obtained by adding the previous two entries, beginning traditionally with 1 and 1. The successive ratios; 2/1, 3/2, 5/3, 8/5, 13/8, ..., converge to the golden ratio Φ.] It is clear from the way in which we solved for Φ that the reciprocal of Φ is $(1/\Phi) = \Phi - 1 = (\sqrt{5}-1)/2 \approx 0.618$. Thus $(1/\Phi) = \sqrt{M}$, and so $(1/M) = \Phi^2 = \Phi + 1$. This means that since M is an exact *discount factor* restriction for period-three cycles, $(1/M) - 1 = \Phi$ is an exact *discount rate* restriction for period-three cycles.

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such that (i) if an optimal program of any dynamic optimization model (Ω, u, δ) exhibits a period-three cycle, then $\delta < M$, and (ii) if $\delta < M$, then it is possible to construct a transition possibility set, Ω, and a reduced-form utility function, u, such that the dynamic optimization model (Ω, u, δ) has an optimal program exhibiting a period-three cycle.[2]

The question stated above has a history which is useful to review in order to place this paper in proper perspective. It stems from the issue of whether chaotic economic behavior can be explained by models of infinitely lived rational agents. Boldrin and Montrucchio [3] took a significant step in addressing this issue when they showed that any twice continuously differentiable function can be a policy function of an appropriately defined dynamic optimization model Thus, in particular, if we take the C^2 function to be the "logistic map" ($h(x) = 4x(1-x)$ for $x \in [0, 1]$), we can show that optimal trajectories can be chaotic for some suitably constructed dynamic optimization model. (This was also independently demonstrated by Deneckere and Pelikan [7]).

It turned out that the dynamic optimization model (Ω, u, δ) constructed to yield the logistic map as its policy function had an extremely small discount factor (about 0.01). This raised the question of whether chaotic trajectories can be ruled out when more "reasonable" discount factors prevailed, or whether this feature of the dynamic optimization model was simply a shortcoming of the particular method used in its construction.

Sorger [21] has constructed examples of dynamic optimization models (using somewhat different methods from those in Boldrin and Montrucchio [3]), for which the logistic map is the policy function when the discount factor is about 0.04. While this is four times larger than the previous discount factors used, it is still very small, entailing a discount rate of over 2000%.

A major step in understanding the relation between discounting and the existence of chaotic optimal trajectories was taken by Sorger [20] when he showed (among other things) that if the policy function of *any* dynamic optimization model is the logistic map, then its associated discount factor must be smaller than 0.5. The remarkable aspect of this discount factor restriction is that it is completely independent of the particular nature of the transition possibility set or of the utility function [beyond the standard properties imposed in such models].

While the result of Sorger [20] settled one issue, it naturally raised another. It was clearly not necessary to generate a logistic map as the

[2] After completing this paper, I learned that essentially the same result has been independently obtained by Nishimura and Yano [18]. Their maintained assumptions on the dynamic optimization model differ somewhat from those used in this paper. Their approach in obtaining the basic result is quite different from mine.

policy function of a dynamic optimization model in order to show that optimal trajectories can be chaotic. Any continuous policy function which generates a period three cycle is chaotic, and the question arises whether the existence of a period three optimal cycle by itself (that is, without knowing other spects of the continuous policy function) leads to suitable restrictions on the discount factor.

Using the theory of stochastic dominance, Sorger [22] showed that if any dynamic optimization model (Ω, u, δ) generates a periodic optimal program of period three, then the discount factor, δ, must satisfy

$$\delta < (\sqrt{5} - 1)/2 \approx 0.618 \tag{1.2}$$

This bound has been later refined to $\delta < 0.5479$ in Sorger [23].

Montrucchio [14] independently arrived at some discount rate restrictions for dynamic complexity of optimal programs by exploiting the theory of strongly concave functions. He established (under a strong concavity assumption on the utility function) a relation between the discount factor of a dynamic optimization model and the topological entropy of its policy function. I have shown elsewhere (see Mitra [13]) that this result can be used to establish the discount factor restriction (1.2) for any dynamic optimization model (Ω, u, δ) which generates a period-three optimal cycle.

The present paper differs from the literature in two significant dimensions. First, we provide an *exact* discount factor restriction, as indicated in (1.1); it also happens to be a *better* restriction than any proposed in the literature so far for period-three cycles.

Second, the method used in this paper is completely distinct from those employed in this literature. It has often been noted that when there is chaotic dynamic behavior, there must be intertemporal arbitrage opportunities for high discount factors, and the discount factor restrictions for optimal chaotic paths is simply a reflection of this fact. However, the methods that have been used in the literature in obtaining these discount factor restrictions do not illustrate this intuition. In contrast, in this paper, we rely heavily on the theory of dual variables or shadow prices associated with optimal programs, and all our discount factor restrictions are obtained from the simple observation that at these "supporting" prices, no activity yields a higher (generalized) profit at any date than the activity chosen along the optimal program at that date. In other words, there are no intertemporal arbitrage opportunities at the prevailing "supporting" prices. The technical advantage of this method for the problem studied in this paper is that it makes all the proofs entirely elementary.

This method, which I refer to as the "value-loss approach" is, of course, very familiar to optimal growth theorists. It is most closely associated with McKenzie's contributions to "turnpike theory" (see McKenzie [11] for a survey). It has also figured prominently in (i) the literature on the stability of Hamiltonian dynamical systems (see especially Cass and Shell [5]) and in (ii) the literature on the intertemporal decentralization of the transversality condition (see especially Brock and Majumdar [4] and Dasgupta and Mitra [6]).

2. Chaos

2a. *Periodic Orbits*

Let I be an interval in $\mathfrak{R}$, the set of reals. Let $f: I \to I$ be a continuous map of the interval I into itself. The pair (I, f) is called a *dynamical system*; I is called the *state space* and f the *law of motion* of the dynamical system.

We write $f^0(x) = x$ and for any integer $n \geqslant 1, f^n(x) = f[f^{n-1}(x)]$. If $x \in I$, the sequence $\tau(x) = \{f^n(x)\}_0^\infty$ is called the *trajectory* from (the initial condition) x. The *orbit* from x is the set $\gamma(x) = \{y: y = f^n(x)$ for some $n \geqslant 0\}$.

A point $x \in I$ is a *fixed point* of f if $f(x) = x$. A point $x \in I$ is called a *periodic point* of f if there is $k \geqslant 1$ such that $f^k(x) = x$. The smallest such k is called the *period* of x. [In particular, if $x \in I$ is a fixed point of f, it is periodic with period 1]. If $x \in I$ is a periodic point with period k, we also say that the orbit of x (or trajectory from x) is periodic with period k.

The following striking result, due to Sarkovskii [19], is a fundamental result on the existence of periodic orbits. A good discussion of this result is contained in Block and Coppel [2].

Proposition 1. *Let the positive integers be totally ordered in the following way:*

$$3 \prec 5 \prec 7 \prec 9 \prec \cdots \prec 2.3 \prec 2.5 \prec \cdots \prec 2^2.3 \prec 2^2.5 \prec \cdots \prec 2^3 \prec 2^2 \prec 2 \prec 1.$$

If f has a periodic orbit of period n and if $n \prec m$, then f also has a periodic orbit of period m.

2b. *Aperiodic Orbits*

In order to study the nature of trajectories which are not periodic, it is useful to define a "scrambled" set. A set $S \subset I$ is called a *scrambled set* if it possesses the following two properties:

(i) If $x, y \in S$ with $x \neq y$, then

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

(ii) If $x \in S$ and y is any periodic point of f,

$$\lim_{n\to\infty} |f^n(x) - f^n(y)| > 0.$$

Thus trajectories starting from points in a scrambled set are not even "asymptotically periodic." Furthermore, for any pair of initial states in the scrambled set, the trajectories move apart and return close to each other infinitely often.

The following theorem, due to Li and Yorke [9], is fundamental in establishing a connection between the existence of period-three cycles and the existence of an uncountable scrambled set.

PROPOSITION 2. *Assume that there is some point x^* in I such that*

$$f^3(x^*) \leqslant x^* < f(x^*) < f^2(x^*) (\text{or } f^3(x^*) \geqslant x^* > f(x^*) > f^2(x^*)). \quad (2.1)$$

Then

(i) *for every positive integer $k = 1, 2, ...,$ there is a periodic point of period k.*

(ii) *there is an uncountable scrambled set $S \subset I$.*

We will say that the dynamical system (I, f) is *chaotic* if conditions (i) and (ii) of Proposition 2 are satisfied.[3] Note that if (I, f) has a periodic point of period three, we can order the three values so that $a < c < b$. Then either (i) $f(c) = b$, in which case $f(b) = a$, and $f(a) = c$, or (ii) $f(c) = a$, in which case $f(a) = b$ and $f(b) = c$. In case (i), choosing $x^* = a$, we have $f(x^*) = f(a) = c > a = x^*$; $f^2(x^*) = f(c) = b > c = f(x^*)$; $f^3(x^*) = f(b) = a = x^*$. In case (ii), choosing $x^* = b$, we have $f(x^*) = f(b) = c < b = x^*$; $f^2(x^*) = f(c) = a < c = f(x^*)$; $f^3(x^*) = f(a) = b = x^*$. Thus, in either case, condition (2.1) of Proposition 2 is satisfied, and (I, f) is chaotic. On the

[3] This is the notion of chaos, which is referred to in the literature as "Li-Yorke chaos." An alternate definition, proposed in the work of Li *et al.* [10] involves the existence of (i) infinitely many periodic points of different periods and (ii) an uncountable scrambled set. They have shown that if f has a periodic point of period which is not a power of 2, then f is chaotic in this sense. We focus, in this paper, on Li-Yorke chaos but our discussion in Section 7 relates to the alternate definition. *Both* definitions involve *topological chaos*, which is quite distinct from the notion of *ergodic chaos*, an important concept in the study of complex dynamic behavior. We do not discuss the concept of ergodic chaos in this paper.

other hand, if (I, f) is chaotic, then by condition (i), it has a periodic point of period three. Thus, (I, f) is chaotic if and only if (I, f) has a periodic point of period three.

3. Dynamic Optimization

3a. *The Model*

The framework is described by a triplet (Ω, u, δ), where Ω, a subset of $\Re_+ \times \Re_+$, is a *transition possibility set*, $u\colon \Omega \to \Re$ is a *utility function* defined on this set, and δ is the *discount factor* satisfying $0 < \delta < 1$.

The transition possibility set describes the states $z \in \Re_+$ to which it is possible to go tomorrow, if one is in state $x \in \Re_+$ today. We define a correspondence $\phi\colon \Re_+ \to \Re_+$ by $\phi(x) = \{y \in R_+ : (x, y) \in \Omega\}$ for each $x \in \Re_+$.

A program $\{x_t\}_0^\infty$ from $\mathbf{x} \in \Re_+$ is a sequence satisfying

$$x_0 = \mathbf{x} \qquad \text{and} \qquad (x_t, x_{t+1}) \in \Omega \qquad \text{for} \quad t \geqslant 0.$$

If one is in state x today and one moves to state z tomorrow (with $(x, z) \in \Omega$) then there is an immediate utility (or "reward" or "return") generated, measured by the utility function, u.

The discount factor, δ, is the weight assigned to tomorrow's utility (compared to today's) in the objective function. The *discount rate* (associated with the discount factor, δ) is given by $\rho = (1/\delta) - 1$.

The following assumptions[4] are imposed on the transition possibility set, Ω, and the utility function, u:

(A.1) (i) $(0, 0) \in \Omega$,(ii) $(0, z) \in \Omega$ implies $z = 0$.

(A.2) Ω is (i) closed, and (ii) convex.

(A.3) There is $\xi > 0$ such that $(x, z) \in \Omega$ and $x \geqslant \xi$ implies $z < x$.

(A.4) If $(x, z) \in \Omega$ and $x' \geqslant x$, $0 \leqslant z' \leqslant z$ then $(x', z') \in \Omega$.

[4] An observation about the set of maintained assumptions, (A.1)–(A.7) is in order at this point. We are concerned with providing an exact discount factor restriction for period-three cycles as indicated in (1.1) in the introduction. This involves two parts, the "necessity" part (i) and the "sufficiency" part (ii). As a general principle, with more assumptions, the sufficiency part becomes harder to establish, the necessity part may become easier. Thus, the choice of the set of maintained assumptions can become an important one. I have chosen to keep the assumptions as close as possible to the "standard" ones used in the literature on intertemporal allocation theory. The reader will note that this has made some assumptions (like the "free-disposal" assumption (A.4) and the "monotonicity" assumption (A.7)) superfluous in providing the "necessity" results (Theorems 1 and 2). They have, however, made the "sufficiency" part somewhat harder to establish.

Clearly, we can pick $0<\zeta<\xi$, such that if $x>\zeta$ and $(x, z) \in \Omega$, then $z<x$.[5] It is straightforward to verify that if $(x, z) \in \Omega$, then $z \leqslant \max(\zeta, x)$. It follows from this that if $\{x_t\}_0^\infty$ is a program from $\mathbf{x} \in \Re_+$, then $x_t \leqslant \max(\zeta, \mathbf{x})$ for $t \geqslant 0$. In particular, if $\mathbf{x} \leqslant \zeta$, then $x_t \leqslant \zeta$ for $t \geqslant 0$. This leads us to choose the closed interval, $[0, \zeta]$ as the natural *state space* of our model, which we will denote by X. We denote the interval $[0, \xi]$ by Y.

The following assumptions are imposed on the utility function, u:

(A.5) u is concave on Ω; further, if (x, z) and (x', z') are in Ω, and $x \neq x'$, then for every $0<\lambda<1$, $u(\lambda(x, z)+(1-\lambda)(x', z'))>\lambda u(x, z)+(1-\lambda) u(x', z')$.

(A.6) u is upper semi-continuous on Ω.

(A.7) If $x, x' \in Y, (x, z) \in \Omega, x' \geqslant x$ and $0 \leqslant z' \leqslant z$, then $u(x', z') \geqslant u(x, z)$.

A program $\{\hat{x}_t\}_0^\infty$ from $\mathbf{x} \geqslant 0$ is an *optimal program* if

$$\sum_0^\infty \delta^t u(x_t, x_{t+1}) \leqslant \sum_0^\infty \delta^t u(\hat{x}_t, \hat{x}_{t+1})$$

for every program $\{x_t\}_0^\infty$ from $\mathbf{x}$.

Under (A.1)–(A.7), there is a unique optimal program from every $\mathbf{x} \in \Re_+$.

3b. *Value and Policy Functions*

The *value function* V: $\Re_+ \rightarrow \Re$ is defined by

$$V(x)=\sum_0^\infty \delta^t u(\hat{x}_t, \hat{x}_{t+1}),$$

where $\{\hat{x}_t\}_0^\infty$ is the optimal program from $x \in \Re_+$.

The *policy function* h: $\Re_+ \rightarrow \Re_+$ is defined by

$$h(x)=\hat{x}_1,$$

where $\{\hat{x}_t\}_0^\infty$ is the optimal program from $x \in \Re_+$.

[5] The fuss about making a distinction between ξ and ζ should perhaps be explained. Clearly ζ can be a "maximum sustainable stock," while ξ cannot. Thus, $Y=[0, \xi]$ is a somewhat larger closed interval than the state space $X=[0, \zeta]$, where the important dynamics will take place. We wanted to make assumption (A.7) on the monotone nature of the utility function in "standard" form; that is, without restricting x, x' in any way. But this created problems in the construction of the example in Section 6b. So we settled for (A.7) in its present form, noting thereby that the monotone restriction on u can be maintained on a larger closed interval than the state space X, while preserving the example in Section 6b. In fact, the reader will note that the method followed in constructing the example allows us to maintain the monotone restriction on u on any large finite interval $[0, \xi']$ containing the state space, by suitable modification in the definition of u in the example.

The properties of the value and policy functions can be summarized in the following result. This is based on Dutta and Mitra [8] and Stokey *et al.* [24].

PROPOSITION 3. (i) *The value function V is strictly concave*[6] *and continuous on $\Re_+$ and non-decreasing on Y. Further, V is the unique continuous function on $Y \equiv [0, \xi]$ which satisfies the functional equation of dynamic programming*:

$$V(x) = \max_{y \in \phi(x)} [u(x, y) + \delta V(y)].$$

(ii) *The policy function h satisfies the following property*: *for each $x \in \Re_+$, $h(x)$ is the unique solution to the constrained maximization problem*:

$$\text{maximize} \quad u(x, y) + \delta V(y)$$

$$\text{subject to} \quad y \in \phi(x).$$

Further, h is continuous on $\Re_+$.

Remarks. (i) In view of the definition of the policy function h, the optimal program from $x \in X$ is the trajectory $\{h^t(x)\}_0^\infty$ generated by the policy function. Thus, an optimal program from $x \in X$ can be called periodic (with period k) if x is a periodic point of h (with period k).

(ii) Since V is concave on $\Re_+$, it has well-defined left-hand and right-hand derivatives for all $x > 0$, which we denote by $\mu(x)$ and $v(x)$ respectively. For $x = 0$, the right-hand derivative $v(0)$ can be defined, but it may be infinite. If $x > y > 0$, then we have $\mu(y) \geqslant v(y) > \mu(x) \geqslant v(x)$, the strict inequality following from the strict concavity of V.

3c. *Price Characterization of Optimality*

Optimality can be conveniently characterized in terms of dual variables or shadow prices. The theory, describing this characterization, is well known and a cornerstone of the theory of optimal economic growth. We state the result which we will use in the following section; a full discussion can be found in Weitzman [26] and McKenzie [11].

[6] The strict concavity of the value function is essential to this study (as it is in the literature discussed in Section 1). Assumption (A.5) is sufficient for this, but is not necessary (see Montrucchio and Sorger [15] for a useful discussion). We have still maintained (A.5), since we view, for this model, the utility function, u, as a "primitive," and the value function, V, as a "derived" concept.

PROPOSITION 4. (a) *If* $\{x_t\}_0^\infty$ *is an optimal program from* $\mathbf{x} \in X$ *and* $\mathbf{x} > 0$, *and there is some* $(\bar{x}, \bar{y}) \in \Omega$ *with* $\bar{y} > 0$ *then there is a sequence* $\{p_t\}_0^\infty$ *of non-negative prices such that for* $t \geqslant 0$,

(i) $\delta^t V(x_t) - p_t x_t \geqslant \delta^t V(x) - p_t x$ *for all* $x \geqslant 0$

(ii) $\delta^t u(x_t, x_{t+1}) + p_{t+1} x_{t+1} - p_t x_t \geqslant \delta^t u(x, y) + p_{t+1} y - p_t x$ *for all* $(x, y) \in \Omega$

(iii) $\lim_{t \to \infty} p_t x_t = 0$.

(b) *If* $\{x_t\}_0^\infty$ *is a program from* $\mathbf{x} \geqslant 0$, *and there is a sequence* $\{p_t\}_0^\infty$ *of non-negative prices such that for* $t \geqslant 0$, (ii) *and* (iii) *above are satisfied, then* $\{x_t\}_0^\infty$ *is an optimal program from* $\mathbf{x}$.

We should add a remark here regarding the proof of part (a) of Proposition 4. If we follow McKenzie (11, Lemma 4.1), we can obtain a sequence $\{p_t\}_0^\infty$, with $p_t \in \mathfrak{R}$ for $t \geqslant 0$, such that (i) and (ii) of Proposition 4(a) are satisfied.[7] Since $\mathbf{x} \in X$ we have $x_t \leqslant \zeta$ for $t \geqslant 0$. By choosing $\zeta < x' \leqslant \xi$, and noting that V is non-decreasing on Y, $V(x') \geqslant V(x_t)$ for $t \geqslant 0$. Now, using (i), $p_t(x' - x_t) \geqslant \delta^t[V(x') - V(x_t)] \geqslant 0$, so that $p_t \geqslant 0$ for $t \geqslant 0$. The "transversality condition" (iii) can be obtained from (i) by choosing $x = (x_t/2)$, so that $0 \leqslant (1/2) p_t x_t \leqslant \delta^t[V(x_t) - V(x_t/2)]$. Since V is continuous on X, there is $m > 0$ such that $|V(x)| \leqslant m$ for all $x \in X$. Since $x_t \in X$ and $(x_t/2) \in X$, $\delta^t[V(x_t) - V(x_t/2)] \leqslant 2\delta^t m$. Since $2\delta^t m \to 0$ as $t \to \infty$, we can conclude that (iii) holds.

If $\{x_t\}_0^\infty$ is a program from $\mathbf{x} \geqslant 0$, and $\{p_t\}_0^\infty$ is a non-negative sequence of prices satisfying (i), (ii), and (iii) of Proposition 4(a), we will say that the program $\{x_t\}_0^\infty$ is *price supported* by $\{p_t\}_0^\infty$.

COROLLARY 1. *If* $\{x_t\}_0^\infty$ *is a periodic optimal program of period* $k \geqslant 2$, *then there is a sequence* $\{q_t\}_0^\infty$ *which price supports* $\{x_t\}_0^\infty$ *such that* $(q_t/\delta^t) = (q_{t-k}/\delta^{t-k})$ *for all* $t \geqslant k$.

Proof. Let $\{x_t\}_0^\infty$ be a periodic optimal program of period $k \geqslant 2$. Then $x_t > 0$, and $x_t \in X$ for $t \geqslant 0$, so Proposition 4 is applicable. Let $\{p_t\}_0^\infty$ be a price support of $\{x_t\}_0^\infty$. Define the "current value" prices $P_t = (p_t/\delta^t)$ for $t \geqslant 0$.

[7] In fact this is all that is needed in proving Theorem 1 (in Section 5) and Theorem 2 (in Section 6). The non-negativity of prices and the necessity of the transversality condition for optimal programs are not needed for that. However, given our maintained set of assumptions (see footnote 2), these properties do follow easily and are noted to keep our exposition closely related to the literature on "price characterizations" of optimal programs.

One can show that a price support $\{q_t\}_0^\infty$ of $\{x_t\}_0^\infty$ can be found such that its "current value" prices, $Q_t = (q_t/\delta^t)$ for $t \geqslant 0$, satisfy the condition

$$Q_t = Q_{t-k} \qquad \text{for} \quad t \geqslant k$$

This can be done by following the method of Sutherland [25] and McKenzie [11]. Since V is continuous on X, there is $m > 0$ such that $|V(x)| \leqslant m$ for all $x \in X$. Denote $\min[x_0, ..., x_{k-1}]$ by ε. Then $\varepsilon > 0$, and by choosing $x = (\varepsilon/2)$ in Proposition 4(i),

$$2m \geqslant V(x_t) - V(\varepsilon/2) \geqslant P_t(\varepsilon/2)$$

so that $0 \leqslant P_t \leqslant (4m/\varepsilon) = m'$ for all $t \geqslant 0$.

Given $j \in [0, 1, 2, ..., k]$, denote for $N \geqslant 0$,

$$Q_j(N) \equiv (P_j + P_{j+k} + P_{j+2k} + \dots + P_{j+Nk})/(N+1).$$

Note that $0 \leqslant Q_j(N) \leqslant m'$ for all j and N. Thus, there is a subsequence $\{N_i\}$, $i = 1, 2, \dots$ such that for $j \in [0, 1, ..., k]$, $Q_j(N_i) \to Q_j$ as $i \to \infty$.

Given $j \in [0, 1, 2, ..., k-1]$, we can use Proposition 4 to get

$$V(x_j) - P_j x_j \geqslant V(x) - P_j x \qquad \text{for all } x \geqslant 0$$

$$u(x_j, x_{j+1}) + \delta P_{j+1} x_{j+1} - P_j x_j \geqslant u(x, z) + \delta P_{j+1} z - P_j x$$
$$\text{for all} \quad (x, z) \in \Omega.$$

The same inequality holds if (P_{j+1}, P_j) are replaced by (P_{j+1+Nk}, P_{j+Nk}) for all $N \geqslant 0$ since $(x_j, x_{j+1}) = (x_{j+Nk}, x_{j+1+Nk})$. Thus, we get, by averaging over the first $(N+1)$ inequalities,

$$V(x_j) - Q_j(N)\, x_j \geqslant V(x) - Q_j(N)\, x \qquad \text{for all} \quad x \geqslant 0$$

$$u(x_j, x_{j+1}) + \delta Q_{j+1}(N)\, x_{j+1} - Q_j(N)\, x_j \geqslant u(x, z) + \delta Q_{j+1}(N)\, z - Q_j(N)\, x$$
$$\text{for all} \quad (x, z) \in \Omega.$$

Letting $N_i \to \infty$, we get

$$V(x_j) - Q_j x_j \geqslant V(x) - Q_j x \qquad \text{for all} \quad x \geqslant 0$$

$$u(x_j, x_{j+1}) + \delta Q_{j+1} x_{j+1} - Q_j x_j \geqslant u(x, z) + \delta Q_{j+1} z - Q_j x$$
$$\text{for all } (x, z) \in \Omega.$$

Now, note that for $N \geqslant 0$,

$$Q_k(N) = [P_k + P_{2k} + \cdots + P_{(N+1)k}]/(N+1)$$
$$= Q_0(N) + [(P_{(N+1)k} - P_0)/(N+1)].$$

Thus we have

$$Q_k = \lim_{i \to \infty} Q_k(N_i) = \lim_{i \to \infty} Q_0(N_i) = Q_0.$$

Now, defining $Q_t = Q_{t-k}$ for $t > k$, and $q_t = Q_t \delta^t$ for $t \geqslant 0$, it is straightforward to check that $\{q_t\}_0^\infty$ is a price support of $\{x_t\}_0^\infty$.

4. The Value-Loss Approach to Minimum Impatience Results

In this section, we describe our approach to obtaining minimum impatience results. It is based on the observation that at the prices supporting an optimal program, there is no activity which yields a higher "generalized profit" at any date (value of utility plus value of terminal stocks minus value of initial stocks at that date) than the activity chosen along the optimal program at that date In other words, there are no arbitrage possibilities available at the supporting prices.

We now proceed to explain this more formally. Let $\{x_t\}_0^\infty$ be an optimal program with supporting prices $\{p_t\}_0^\infty$. Then, the *value loss* at date t if the economy chooses $(x, z) \in \Omega$ (instead of $(x_t, x_{t+1}) \in \Omega$) is

$$L_t(x, z; \{x_t, p_t\}_0^\infty) = \delta^t u(x_t, x_{t+1}) + p_{t+1} x_{t+1} - p_t x_t - [\delta^t u(x, z) + p_{t+1} z - p_t x]. \tag{4.1}$$

Since $\{p_t\}_0^\infty$ supports $\{x_t\}_0^\infty$, we have

$$L_t(x, z; \{x_t, p_t\}_0^\infty) \geqslant 0 \qquad \text{for all} \quad (x, z) \in \Omega. \tag{4.2}$$

Now let $\{y_t\}_0^\infty$ be another optimal program with supporting prices $\{q_t\}_0^\infty$. Then the value loss at date t if the economy chooses $(x, z) \in \Omega$ (instead of $(y_t, y_{t+1}) \in \Omega$) is

$$L_t(x, z; \{y_t, q_t\}_0^\infty) = \delta^t u(y_t, y_{t+1}) + q_{t+1} y_{t+1} - q_t y_t - [\delta^t u(x, z) + q_{t+1} z - q_t x]. \tag{4.3}$$

Since $\{q_t\}_0^\infty$ supports $\{y_t\}_0^\infty$, we have

$$L_t(x, z; \{y_t, q_t\}_0^\infty) \geqslant 0 \qquad \text{for all} \quad (x, z) \in \Omega. \tag{4.4}$$

If we apply (4.1) to $(y_t, y_{t+1}) \in \Omega$, we get

$$\delta^t u(x_t, x_{t+1}) + p_{t+1} x_{t+1} - p_t x_t = \delta^t u(y_t, y_{t+1}) + p_{t+1} y_{t+1} - p_t y_t + L_t(y_t, y_{t+1}, \{x_t, p_t\}_0^\infty). \tag{4.5}$$

Similarly, if we apply (4.3) to $(x_t, x_{t+1}) \in \Omega$, we get

$$\delta^t u(y_t, y_{t+1}) + q_{t+1} y_{t+1} - q_t y_t = \delta^t u(x_t, x_{t+1}) + q_{t+1} x_{t+1} - q_t x_t + L_t(x_t, x_{t+1}; \{y_t, q_t\}_0^\infty). \tag{4.6}$$

Combining (4.5) and (4.6), we get

$$L_t(x_t, x_{t+1}; \{y_t, q_t\}_0^\infty) + L_t(y_t, y_{t+1}; \{x_t, p_t\}_0^\infty) = -(p_{t+1} - q_{t+1})(y_{t+1} - x_{t+1}) + (p_t - q_t)(y_t - x_t). \tag{4.7}$$

Using (4.2) and (4.4) in (4.7), we get

$$(p_{t+1} - q_{t+1})(y_{t+1} - x_{t+1}) \leqslant (p_t - q_t)(y_t - x_t). \tag{4.8}$$

Denoting (p_t/δ^t) by P_t and (q_t/δ^t) by Q_t for $t \geqslant 0$ (these are, of course, the "current value prices"), we get

$$\delta(P_{t+1} - Q_{t+1})(y_{t+1} - x_{t+1}) \leqslant (P_t - Q_t)(y_t - x_t). \tag{4.9}$$

By (A.5), the inequality in (4.9) is strict if $x_t \neq y_t$.

This fundamental inquality yields *all* the discount factor restrictions that we will derive in this paper. But, before we do, we need to know the sign of the typical term $(P_t - Q_t)(y_t - x_t)$. This is fairly easy. Since $\{p_t\}_0^\infty$ is a price support of $\{x_t\}_0^\infty$,

$$\delta^t V(x_t) - p_t x_t \geqslant \delta^t V(y_t) - p_t y_t. \tag{4.10}$$

Since $\{q_t\}_0^\infty$ is a price support of $\{y_t\}_0^\infty$,

$$\delta^t V(y_t) - q_t y_t \geqslant \delta^t V(x_t) - q_t x_t. \tag{4.11}$$

Combining (4.10) and (4.11),

$$(p_t - q_t)(y_t - x_t) \geqslant 0. \tag{4.12}$$

Dividing through in (4.12) by δ^t,

$$(P_t - Q_t)(y_t - x_t) \geqslant 0 \tag{4.13}$$

By the strict concavity of V, the inequality in (4.13) is strict if $y_t \neq x_t$. We summarize the above discussion in the following proposition.

PROPOSITION 5. *Let (Ω, u, δ) be a dynamic optimization model. Suppose $\{x_t\}_0^\infty$ is an optimal program with price support $\{p_t\}_0^\infty$, and $\{y_t\}_0^\infty$ is an* *optimal program with price support* $\{q_t\}_0^\infty$. *Denoting* (p_t/δ^t) *by* P_t *and* (q_t/δ^t) *by* Q_t *for* $t \geqslant 0$ *we have*

(i) $\delta(P_{t+1} - Q_{t+1})(y_{t+1} - x_{t+1}) \leqslant (P_t - Q_t)(y_t - x_t)$ *for* $t \geqslant 0$

(ii) $(P_t - Q_t)(y_t - x_t) \geqslant 0$ *for* $t \geqslant 0$.

Furthermore, if $y_t \neq x_t$ *from some* t, *then the inequalities in* (i) *and* (ii) *are strict for that* t.

Let us demonstrate how the basic inequalities of Proposition 5 can be used to obtain minimum impatience theorems. The typical situation that we will be applying it to will involve an optimal program $\{x_t\}_0^\infty$ exhibiting the pattern

$$x_2 < x_0 < x_1.$$

That is, first the state variable increases, $(x_1 > x_0)$ and then it falls below the *original level* $(x_2 < x_0)$. We will show[8] that this immediately leads to a discount factor restriction

$$\delta < (x_1 - x_0)/(x_1 - x_2). \tag{4.14}$$

PROPOSITION 6. *Let* (Ω, u, δ) *be a dynamic optimization model. Suppose* $\{x_t\}_0^\infty$ *is an optimal program.*

(i) *If* $x_2 < x_0 < x_1$, *then* $\delta < (x_1 - x_0)/(x_1 - x_2)$.

(ii) *If* $x_1 < x_0 < x_2$, *then* $\delta < (x_0 - x_1)/(x_2 - x_1)$.

Proof. We will only establish (i), since (ii) follows by similar arguments.

If $x_2 < x_0 < x_1$, then $x_0 > 0$, and $(x_0, x_1) \in \Omega$ with $x_1 > 0$. Thus, Proposition 4 is applicable, and let $\{p_t\}_0^\infty$ be a price support of $\{x_t\}_0^\infty$. The sequence $\{y_t\}_0^\infty$ defined by $y_t = x_{t+1}$ for $t \geqslant 0$ is a program from $y_0 \equiv x_1$, and it is clearly an optimal program from y_0. Defining a sequence $\{q_t\}_0^\infty$ by $q_t = (p_{t+1}/\delta)$ for $t \geqslant 0$. it is straightforward to check that $\{q_t\}_0^\infty$ is a price support of $\{y_t\}_0^\infty$. Thus, Proposition 5 is applicable.

Applying Proposition 5 for $t = 0$, and noting that $x_0 \neq x_1 = y_0$, $x_1 \neq x_2 = y_1$, we get

$$0 < \delta(P_1 - Q_1)(y_1 - x_1) < (P_0 - Q_0)(y_0 - x_0). \tag{4.15}$$

[8] Although Proposition 6 has not been explicitly noted in the literature, Gerhard Sorger has used it in his oral presentations on the subject, to show how just three observations on an (assumed) optimal program might yield a rather good upper bound on the discount factor.

This yields the discount factor restriction

$$\delta < (P_0 - Q_0)(y_0 - x_0)/(Q_1 - P_1)(x_1 - y_1). \tag{4.16}$$

It follows from Proposition 4(i) that $v(x_t) \leqslant P_t$, $v(y_t) \leqslant Q_t$; furthermore $P_t \leqslant \mu(x_t)$, $Q_t \leqslant \mu(y_t)$ whenever $x_t > 0$, $y_t > 0$. Thus, we get

$$P_0 \geqslant v(x_0) > \mu(x_1) = \mu(y_0) \geqslant Q_0 \tag{4.17}$$

since $x_0 < x_1$ and V is strictly concave. Similarly, we get

$$Q_1 \geqslant v(y_1) = v(x_2) > \mu(x_1) \geqslant P_1 \tag{4.18}$$

since $x_2 < x_1$ and V is strictly concave. So $(P_0 - Q_0) > 0$, $(Q_1 - P_1) > 0$ and

$$(Q_1 - P_1) \geqslant [v(x_2) - P_1] > [\mu(x_0) - P_1] \geqslant [P_0 - P_1] = (P_0 - Q_0). \tag{4.19}$$

Also $(y_0 - x_0) = (x_1 - x_0) > 0$, and $(x_1 - y_1) = (x_1 - x_2) > 0$. Thus, (4.16) yields

$$\delta < \frac{(y_0 - x_0)}{(x_1 - y_1)} = \frac{(x_1 - x_0)}{(x_1 - x_2)},$$

which establishes the result.

Remark. Proposition 6 imposes strong discount factor restrictions on dynamic optimization models yielding the logistic map or the tent map as their policy functions.

If a dynamic optimization model (Ω, u, δ) has the tent map ($h(x) = 2x$ for $0 \leqslant x \leqslant 0.5$, $h(x) = 2 - 2x$ for $0.5 \leqslant x \leqslant 1$) as its policy function, then $(x_0, x_1, x_2 ...) = (1/2, 1, 0, 0, ...)$ is an optimal program with $x_2 < x_0 < x_1$, so that δ must be less than (1/2) by Proposition 6.

If (Ω, u, δ) has the logistic map ($h(x) = 4x(1 - x)$ for $0 \leqslant x \leqslant 1$) as its policy function, then $(x_0, x_1, x_2, ...) = (0.625, 0.9375, 0.234375, ...)$ is an optimal program with $x_2 < x_0 < x_1$, so that $\delta < 0.4$ by Proposition 6. More generally, if (Ω, u, δ) has the quadratic map ($h(x) = \mu x(1 - x)$ for $0 \leqslant x \leqslant 1$, with $3 < \mu \leqslant 4$) as its policy function, then choosing $x_0 = 0.5(1 + \mu)/\mu$, $x_1 = h(x_0)$, $x_2 = h(x_1)$, one can show that $\delta < 4/(\mu - 1)^2$ by applying Proposition 6.

5. A Discount Factor Restriction for Period-Three Cycles

While Proposition 6 is useful in obtaining some discount factor restrictions (as noted in the previous section) its shortcoming is that it may not be very useful when the only information we have is that there is a period-three cycle (i.e., when the nature of the policy function on the rest of its domain is not known).

Proceeding more formally, suppose $\{x_t\}_0^\infty$ is an optimal program of a dynamic optimization model (Ω, u, δ), exhibiting a period-three cycle. Then we can order the set $\{x_0, x_1, x_2\}$ on the real line. Let the highest value be b, the lowest value be a, and the middle value be c. Then, denoting the policy function by h, we have two possibilities (i) $h(c) = a$, (ii) $h(c) = b$. In (i), we must also have $h(a) = b$ and $h(b) = c$; similarly in case (ii), we must have $h(b) = a$ and $h(a) = c$. Thus, denoting the optimal program from c by $\{y_t\}_0^\infty$, we have two possibilities: (i) $y_2 < y_0 < y_1$, or (ii) $y_1 < y_0 < y_2$. Then Proposition 6 is applicable, yielding

$$\delta < (y_1 - y_0)/(y_1 - y_2). \tag{5.1}$$

But, not knowing exactly where y_0 is between y_1 and y_2, the discount factor restriction obtained from (5.1) can be quite weak (when y_0 is close to y_2).

The way to strengthen the restriction imposed by (5.1) is suggested by the observation that Proposition 6 does not fully exploit the fact that there is a period-three cycle. In particular, $y_3 = h(y_2) = y_0$ is nowhere used, and this information should yield a restriction in addition to (5.1). This additional restriction turns out to be

$$\delta^2 < (y_0 - y_2)/(y_1 - y_2). \tag{5.2}$$

Now the two restrictions can be combined to yield

$$\delta < [\sqrt{5} - 1]/2, \tag{5.3}$$

which is completely independent of the actual value of y_0 (or, for that matter, y_1 and y_2). This is formally demonstrated in the following theorem.

THEOREM 1. *Let (Ω, u, δ) be a dynamic optimization model, with a policy function h. If h exhibits a period-three cycle, then*

$$\delta < [\sqrt{5} - 1]/2. \tag{5.4}$$

Proof. Let the three values of the period-three cycle be ordered so that $a < c < b$. There are then exactly two possibilites: (i) $h(c) = b$ [in which case $h(b) = a$ and $h(a) = c$], or (ii) $h(c) = a$ [in which case $h(a) = b$ and $h(b) = c$]. We concentrate on the first possibility, since the second one can be analyzed similarly.

In case (i), let $\{x_t\}_0^\infty$ be the optimal program from c. Then $x_0 = c$, $x_1 = b$, $x_2 = a$, $x_3 = c$, and

$$x_2 < x_0 < x_1.$$

Applying Proposition 6, we have

$$\delta < (x_1 - x_0)/(x_1 - x_2). \tag{5.5}$$

Note that $x_0 > 0$, and $(x_0, x_1) \in \Omega$ with $x_1 > 0$, so Proposition 4 is applicable, and there is a sequence $\{p_t\}_0^\infty$ which provides a price support to $\{x_t\}_0^\infty$.

The sequence $\{y_t\}_0^\infty$ defined by $y_t = x_{t+2}$ for $t \geqslant 0$ is a program from $y_0 = x_2$, and it is clearly an optimal program from y_0. Defining $\{q_t\}_0^\infty$ by $q_t = (p_{t+2}/\delta^2)$ for $t \geqslant 0$, it is easy to check that $\{q_t\}_0^\infty$ is a price support of $\{y_t\}_0^\infty$. Applying Proposition 5 for $t = 0$, and noting that $x_0 \neq x_2 = y_0$, $x_1 \neq x_0 = x_3 = y_1$, we get

$$0 < \delta(P_1 - Q_1)(y_1 - x_1) < (P_0 - Q_0)(y_0 - x_0). \tag{5.6}$$

Applying Proposition 5 for $t = 1$, and noting that $x_1 \neq x_0 = y_1$, $x_2 \neq x_1 = x_4 = y_2$, we get

$$0 < \delta(P_2 - Q_2)(y_2 - x_2) < (P_1 - Q_1)(y_1 - x_1). \tag{5.7}$$

Combining (5.6) and (5.7), we get

$$\delta^2 < \frac{(P_0 - Q_0)(y_0 - x_0)}{(P_2 - Q_2)(y_2 - x_2)} = \frac{(Q_0 - P_0)(x_0 - y_0)}{(P_2 - Q_2)(y_2 - x_2)} = \frac{(P_2 - P_0)(x_0 - y_0)}{(P_2 - Q_2)(y_2 - x_2)}. \tag{5.8}$$

It follows from Proposition 4(i) that

$$P_0 \leqslant \mu(x_0) < v(x_2) \leqslant P_2 \tag{5.9}$$

since $x_0 > x_2$ and V is strictly concave. Similarly, we get

$$P_2 \geqslant v(x_2) > \mu(x_1) = \mu(y_2) \geqslant Q_2 \tag{5.10}$$

since $x_2 < x_1$ and V is strictly concave. Thus $(P_2 - P_0) > 0$ and $(P_2 - Q_2) > 0$, and

$$(P_2 - Q_2) \geqslant [P_2 - \mu(y_2)] = [P_2 - \mu(x_1)] > [P_2 - v(x_0)] \geqslant (P_2 - P_0). \tag{5.11}$$

Thus (5.8) yields the restriction

$$\delta^2 < (x_0 - y_0)/(y_2 - x_2) = (x_0 - x_2)/(x_1 - x_2). \tag{5.12}$$

Denote $(x_1 - x_0)/(x_1 - x_2)$ by α. Then $0 < \alpha < 1$, and $(1 - \alpha) = (x_0 - x_2)/(x_1 - x_2)$. Thus using (5.5) and (5.12),

$$\delta < \min[\alpha, \sqrt{1 - \alpha}]. \tag{5.13}$$

Denote $(\sqrt{5}-1)/2$ by β, so that $\beta^2=(1-\beta)$. Thus if $\alpha \geqslant \beta$, then $\min[\alpha, \sqrt{1-\alpha}] \leqslant \sqrt{1-\alpha} \leqslant \sqrt{1-\beta}=\beta$. And if $0<\alpha<\beta$, then $\min[\alpha, \sqrt{1-\alpha}]<\beta$. Thus in either case,

$$\min[\alpha, \sqrt{1-\alpha}] \leqslant \beta=(\sqrt{5}-1)/2. \tag{5.14}$$

Combining (5.13) and (5.14), we get (5.4).

Remark. Sorger [22] obtained the discount factor restriction (5.4) of Theorem 1, by applying the theory of stochastic dominance. The restriction has been refined to $\delta<0.5479$ in Sorger [23], using the same methods.

Montrucchio [14] does not address directly the problem of finding a discount factor restriction for period three cycles. However, he establishes (under strong concavity assumptions on the utility function) that if (Ω, u, δ) is a dynamic optimization model with policy function h, and the topological entropy of h is $\Psi(h)$, then the discount factor, δ, is related to the topological entropy by the inequality

$$\delta \leqslant e^{-\Psi(h)}. \tag{5.15}$$

It has been shown in Mitra [13] that if h exhibits a period-three cycle then (5.15) can be used to obtain the discount factor restriction (5.4) of Theorem 1. The inequality (5.15) actually holds under the assumptions used in this paper: this follows from the recent work of Montrucchio and Sorger [15].

6. An Exact Discount Factor Restriction for Period-Three Cycles

6a. *The Main Result*

Theorem 1 provides a strong restriction on the discount factor for period-three cycles, but as we mentioned in the previous section, stronger restrictions have already been obtained. This suggests that, in establishing the Theorem, we have not exploited the full implication of the existence of a period-three cycle; in other words, we have thrown out useful information somewhere along the line of our argument.

In order to see most clearly what we have left out, assume that the value function is differentiable at the points of the period-three cycle. Then the "current value prices" are simply the derivatives of the value function at the respective points.

Now, if we look at the proof of Theorem 1 carefully, we notice that the restriction (5.5) is obtained from the inequality (4.16) in Proposition 6 which now reads

$$\delta < \frac{[V'(x_0) - V'(x_1)](x_1 - x_0)}{[V'(x_2) - V'(x_1)](x_1 - x_2)}. \tag{6.1}$$

Similarly, the restriction (5.12) is obtained from the inequality (5.8), which reads

$$\delta^2 < \frac{[V'(x_2) - V'(x_0)](x_0 - x_2)}{[V'(x_2) - V'(x_1)](x_1 - x_2)}. \tag{6.2}$$

We know that $[V'(x_0) - V'(x_1)]/[V'(x_2) - V'(x_1)]$ is between 0 and 1 and thus (6.1) yields (5.5). But nothing more is known about the ratio, and it could well be very close to 1. The same could be said about the ratio $[V'(x_2) - V'(x_0)]/[V'(x_2) - V'(x_1)]$ and the relation of (6.2) to (5.12). But *both* ratios cannot be close to 1: in fact, if one is close to 1, the other is close to zero, independent of how concave V is. In this regard, these ratios play exactly the same role in the expressions (6.1) and (6.2), as the ratios involving the primal variables $(x_1 - x_0)/(x_1 - x_2)$ and $(x_0 - x_2)/(x_1 - x_2)$. Since we were able to exploit the inequalities (5.5) and (5.12), involving only the primal variables, to get a discount factor restriction of $(\sqrt{5} - 1)/2$, we should be able to exploit (6.1) and (6.2) to yield a restriction which is exactly the *square* of this number. We now proceed to prove this formally [No assumption regarding the differentiability of V will be made in the formal proof].

THEOREM 2. *Let (Ω, u, δ) be a dynamic optimization model, with a policy function h. If h exhibits a period-three cycle then*

$$\delta < [(\sqrt{5} - 1)/2]^2. \tag{6.3}$$

Proof. As in the proof of Theorem 1, let the three values of the period-three cycle be ordered so that $a < c < b$. There are two possibilities to consider: (i) $h(c) = b$ [in which case $h(b) = a$ and $h(a) = c$], or (ii) $h(c) = a$ [in which case $h(a) = b$ and $h(b) = c$]. We concentrate on possibility (i); (ii) can be analyzed similarly.

In case (i), let $\{x_t\}_0^\infty$ be the optimal program from c. Then $x_0 = c$, $x_1 = b$, $x_2 = a$, $x_3 = c$, and $x_2 < x_0 < x_1$. Let $\{p_t\}_0^\infty$ be a price support of $\{x_t\}_0^\infty$.

The sequence $\{z_t\}_0^\infty$ defined by $z_t = x_{t+1}$ for $t \geqslant 0$ is a program from $z_0 = x_1$, and it is the optimal program from z_0. Defining a sequence $\{r_t\}_0^\infty$

by $r_t = (p_{t+1}/\delta)$ for $t \geqslant 0$, it is easy to check that $\{r_t\}_0^\infty$ is a price support of $\{z_t\}_0^\infty$. Denote (p_t/δ_t) by P_t and (r_t/δ^t) by R_t for $t \geqslant 0$.

Now, following the proof of Proposition 6, we get

$$\delta < \frac{(P_0 - R_0)(z_0 - x_0)}{(R_1 - P_1)(x_1 - z_1)}, \tag{6.4}$$

where $0 < (P_0 - R_0) < (R_1 - P_1)$, and $0 < (z_0 - x_0) < (x_1 - z_1)$. Now $R_0 = r_0 = (p_1/\delta) = P_1$, and $R_1 = (r_1/\delta) = (P_2/\delta^2) = P_2$. Also $z_0 = x_1$ and $z_1 = x_2$. Thus (6.4) can be rewritten as

$$\delta < \frac{(P_0 - P_1)(x_1 - x_0)}{(P_2 - P_1)(x_1 - x_2)}. \tag{6.5}$$

The sequence $\{y_t\}_0^\infty$ defined by $y_t = x_{t+2}$ for $t \geqslant 0$ is a program from $y_0 = x_2$, and it is the optimal program from y_0. Defining a sequence $\{q_t\}_0^\infty$ by $q_t = (p_{t+2}/\delta^2)$ for $t \geqslant 0$, it is easy to check that $\{q_t\}_0^\infty$ is a price support of $\{y_t\}_0^\infty$. Denote (q_t/δ^t) by Q_t for $t \geqslant 0$.

Then, following the proof of Theorem 1, we get

$$\delta^2 < \frac{(Q_0 - P_0)(x_0 - y_0)}{(P_2 - Q_2)(y_2 - x_2)}, \tag{6.6}$$

where $0 < (Q_0 - P_0) < (P_2 - Q_2)$, and $0 < (x_0 - y_0) < (y_2 - x_2)$. Now, $Q_0 = q_0 = (p_2/\delta^2) = P_2$ and $Q_2 = (q_2/\delta^2) = (p_4/\delta^4) = P_4 = P_1$, the last equality following from Corollary 1. Also $y_0 = x_2$ and $y_2 = x_1$. Thus (6.6) can be rewritten as

$$\delta^2 < \frac{(P_2 - P_0)(x_0 - x_2)}{(P_2 - P_1)(x_1 - x_2)}. \tag{6.7}$$

Denote $(P_0 - P_1)/(P_2 - P_1)$ by α; then $0 < \alpha < 1$, and $(1 - \alpha) = (P_2 - P_0)/(P_2 - P_1)$. Denote $(x_1 - x_0)/(x_1 - x_2)$ by γ; then $0 < \gamma < 1$ and $(1 - \gamma) = (x_0 - x_2)/(x_1 - x_2)$. Thus (6.5) and (6.7) yield the restriction

$$\delta < \min[\alpha\gamma, \sqrt{(1-\alpha)(1-\gamma)}]. \tag{6.8}$$

Denoting $(\alpha + \gamma)/2$ by η, we have

$$\min[\alpha\gamma, \sqrt{(1-\alpha)(1-\gamma)}] \leqslant \min[\eta^2, \sqrt{(1-\eta)^2}]. \tag{6.9}$$

Using (6.9) in (6.8), we get

$$\delta < \min[\eta^2, (1-\eta)]. \tag{6.10}$$

As in the proof of Theorem 1, denote $(\sqrt{5}-1)/2$ by β, so that $\beta^2 = 1-\beta$. If $\eta \geqslant \beta$, then $(1-\eta) \leqslant (1-\beta) = \beta^2$, so that $\min[\eta^2, (1-\eta)] \leqslant \beta^2$. If $0 < \eta < \beta$, then $\eta^2 < \beta^2$, so that $\min[\eta^2, (1-\eta)] < \beta^2$, Thus, in either case,

$$\min[\eta^2, (1-\eta)] \leqslant \beta^2. \tag{6.11}$$

Using (6.11) in (6.10), we get (6.3).

6b. *The Example*

We now proceed to show that the discount factor restriction obtained in Theorem 2 for period-three optimal cycles is the best possible. We do this by constructing an example, when $\delta < [(\sqrt{5}-1)/2]^2$, of a transition possibility set, Ω, and a utility function, u, such that (Ω, u, δ) exhibits a period-three optimal cycle.

The constructed example is a simple adaptation, for our purpose, of the example given in Nishimura *et al.* [16]. Denote $[(\sqrt{5}-1)/2]^2$ by M, and assume that $0 < \delta < M$. Define, $\gamma = 1/\sqrt{M}$ so that $1 < \gamma < 2$ and $\gamma^2 M = 1$. Then $\gamma^2\delta < 1$, and it is possible to find $\alpha > 1$, such that $\gamma^2\delta < (1/\alpha)$. Choose $\theta > \alpha\delta/(1-\gamma^2\delta)$ and $\beta > 2\max\{(1/\gamma^2\delta), 2\alpha + \gamma\theta(2\gamma - 1)\}$.

Define the transition possibility set, Ω, by

$$\Omega = \{(x, z) \in \mathfrak{R}^2_+ : z \leqslant \min(\gamma x, 1)\}.$$

Then (A.1)–(A.4) are satisfied.

Let $H_{\leqslant} = \{x, z) \in \mathfrak{R}^2 : z \leqslant 2 - \gamma x\}$ and $H_{\geqslant} = \{(x, z) \in \mathfrak{R}^2 : z \geqslant 2 - \gamma x\}$. Define the reduced form utility function $u: \Omega \to \mathfrak{R}$ by

$$u(x, z) = [\beta - (2/\gamma)]\, x - (\alpha - 1)\, x^2 - [\beta\delta - (2/\gamma^2)]\, z - [(1/\gamma^2) - \alpha\delta]\, z^2$$
$$\text{for} \quad (x, z) \in H_{\leqslant} \cap \Omega$$
$$u(x, z) = (4\gamma\theta + \beta)\, x - (\alpha + \gamma^2\theta)\, x^2 - (\beta\delta - 4\theta)\, z - (\theta - \alpha\delta)\, z^2 - 2\gamma\theta xz - 4\theta$$
$$\text{for} \quad (x, z) \in H_{\geqslant} \cap \Omega.$$

Then, by using the proof of Lemma 3.1 in Nishimura *et al.* [16], (A.5)–(A.7) are satisfied.

By using the proof of Lemma 3.2 of Nishimura *et al.* [16], the policy function of the constructed model (Ω, u, δ) is given by

$$h(x) = \begin{cases} \gamma x & \text{for} \quad x \in [0, 1/\gamma] \\ 2 - \gamma x & \text{for} \quad x \in [1/\gamma, 1]. \end{cases}$$

Note that $h(1) = (2-\gamma)$, and $(2-\gamma) = 2 - (1/\sqrt{M}) = 1 - \sqrt{M} = M$, so that $0 < (2-\gamma) < \sqrt{M} = (1/\gamma)$. Thus $h(2-\gamma) = \gamma(2-\gamma) = (M/\sqrt{M}) = \sqrt{M}$, and $h(\sqrt{M}) = 1$. Thus $(1, M, \sqrt{M}, 1, M, \sqrt{M}, \ldots)$ is the optimal program starting from 1. This optimal program clearly exhibits a period-three cycle.

7. Discount Factor Restrictions for Periodic Optimal Programs

In this final section, we will find discount factor restrictions for any dynamic optimization model exhibiting a periodic optimal program of period not equal to a power of 2. To illustrate our method of deriving these restrictions most transparently, we first obtain a restriction for an important special case.

Specifically, we obtain an upper bound on the discount factor, δ, that must be satisfied in order that a dynamic optimization model (Ω, u, δ) yield a periodic optimal program of odd period greater than one. It is important to note that this bound holds *uniformly* for all odd periods greater than one.

In order to establish these results, we will find it convenient to present a preliminary result which compares the value and policy functions of two dynamic optimization models. To proceed more formally, let us refer to a triplet (Ω, u, δ) satisfying (A.1)–(A.7) as a *dynamic optimization model.* Now, let $\mathfrak{I} = (\Omega, u, \delta)$ be a dynamic optimization model, with value function V and policy function h. We can then construct[9] another dynamic optimization model $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$ with $\delta^* = \delta^2$, such that the policy function, h^*, of $\mathfrak{I}^*$ is h^2 and the value function, V^*, of $\mathfrak{I}^*$ is V.

Proposition 7. *Let $\mathfrak{I} = (\Omega, u, \delta)$ be a dynamic optimization model, with value function V and policy function h. Then, there exists Ω^* and u^* satisfying* (A.1)–(A.7) *such that with $\delta^* = \delta^2$,* (i) *the value function, V^*, of the dynamic optimization model $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$ is given by V, and* (ii) *the policy function, h^*, of $\mathfrak{I}^*$ is given by h^2.*

Proof. Define $\Omega = \{(x, z)$: there is $y \in \Re_+$ such that $(x, y) \in \Omega$ and $(y, z) \in \Omega\}$. Define, next, $u^*: \Omega^* \to \Re$ by

$$\left.\begin{aligned} u^*(x, z) = \max[u(x, y) + \delta u(y, z)] \\ \text{subject to } (x, y) \in \Omega \\ \text{and } (y, z) \in \Omega. \end{aligned}\right\} (P).$$

It is straightforward to check that Ω^* and u^* satisfy assumptions (A.1)–(A.5) and (A.7).

To check (A.6), one essentially applies the method used to obtain a version of a maximum theorem in Berge [1; Theorem 2, p. 116].

[9] Sorger [21] proves a result similar to Proposition 7. However his set of maintained assumptions on a dynamic optimization model is different from ours. It is not clear to us whether Proposition 7 is valid if the upper-semicontinuity of the utility function in (A.6) is replaced by continuity.

Let (x^s, z^s), $s = 1, 2, \ldots$, be a sequence of elements in Ω^* converging to (x^0, z^0). For (x^s, z^s), $s = 1, 2, \ldots$, let y^s solve problem (P). Then since $x^s \to x^0$ as $s \to \infty$, and $(x^s, y^s) \in \Omega$, y^s is in a bounded set (by (A.1) and (A.2)). Let $(y^{s'})$ be an arbitrary convergent subsequence of (y^s), converging to y^0.

Since $(x^{s'}, y^{s'}) \to (x^0, y^0)$, and $(y^{s'}, z^{s'}) \to (y^0, z^0)$ as $s' \to \infty$, and $(x^{s'}, y^{s'}) \in \Omega$ and $(y^{s'}, z^{s'}) \in \Omega$ for all s', we have $(x^0, y^0) \in \Omega$ and $(y^0, z^0) \in \Omega$. Now, we have

$$\begin{aligned}\overline{\lim_{s' \to \infty}}\ u^*(x^{s'}, z^{s'}) &= \overline{\lim_{s' \to \infty}}\ [u(x^{s'}, y^{s'}) + \delta u(y^{s'}, z^{s'})]\\ &\leqslant \overline{\lim_{s' \to \infty}}\ u(x^{s'}, y^{s'}) + \delta\ \overline{\lim_{s' \to \infty}}\ u(y^{s'}, z^{s'})\\ &\leqslant u(x^0, y^0) + \delta u(y^0, z^0),\end{aligned}$$

the last inequality following from the upper semicontinuity of u. By definition of u^*, we also have

$$u(x^0, y^0) + \delta u(y^0, z^0) \leqslant u^*(x^0, z^0).$$

Thus, we obtain

$$\overline{\lim_{s' \to \infty}}\ u^*(x^{s'}, z^{s'}) \leqslant u^*(x^0, z^0).$$

Since $(y^{s'})$ was an arbitrary convergent subsequence of (y^s), this establishes property (A.6) of u^*.

Denote the value and policy functions of the dynamic optimization model $(\Omega^*, u^*, \delta^*)$, where $\delta^* = \delta^2$, by V^* and h^*.

To compare the value functions of the two dynamic optimization models, $\mathfrak{I} = (\Omega, u, \delta)$ and $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$, we will find it convenient to refer to programs and optimal programs in the former by the prefix $\mathfrak{I}$ and in the latter by $\mathfrak{I}^*$.

Let $\mathbf{x} \geqslant 0$ be arbitrarily given, and let $\{x_t\}$ be $\mathfrak{I}^*$-optimal from $\mathbf{x}$. Then there is $\{y_t\}$ such that y_t solves (P) for $(x, z) = (x_t, x_{t+1})$. Define $k_t = x_{t/2}$ for $t = 0, 2, 4, \ldots$, and $k_t = y_{(t-1)/2}$ for $t = 1, 3, 5, \ldots$ Then $\{k_t\}$ is a $\mathfrak{I}$ program. Further,

$$\begin{aligned}V^*(\mathbf{x}) &= \sum_0^\infty (\delta^2)^t\, u^*(x_t, x_{t+1})\\ &= \sum_0^\infty (\delta^2)^t\, [u(x_t, y_t) + \delta u(y_t, x_{t+1})]\\ &= \sum_0^\infty \delta^t\, u(k_t, k_{t+1}) \leqslant V(\mathbf{x}).\end{aligned}$$

On the other hand, let $\{k'_t\}$ be $\mathfrak{I}$-optimal from $\mathbf{x}$. Then, defining $\{x_t\}$ by $x_t = k'_{2t}$ for $t \geqslant 0$, $\{x_t\}$ is a $\mathfrak{I}^*$-program from $\mathbf{x}$, and

$$\begin{aligned} V(\mathbf{x}) &= \sum_0^\infty \delta^t u(k'_t, k'_{t+1}) \\ &= \sum_0^\infty (\delta^2)^t \left[u(k'_{2t}, k'_{2t+1}) + \delta u(k'_{2t+1}, k'_{2t+2})\right] \\ &\leqslant \sum_0^\infty (\delta^2)^t u^*(k'_{2t}, k'_{2t+2}) \leqslant V^*(\mathbf{x}). \end{aligned}$$

Thus, $V(\mathbf{x}) = V^*(\mathbf{x})$, which shows that $V = V^*$, since $\mathbf{x} \geqslant 0$ was arbitrary.

To compare the policy functions, let $\mathbf{x} \geqslant 0$ be arbitrary, and let $\{k_t\}$ be $\mathfrak{I}$-optimal from $\mathbf{x}$. Then defining $x_t = k_{2t}$ for $t \geqslant 0$, $\{x_t\}$ is a $\mathfrak{I}$-program from $\mathbf{x}$. Now,

$$\begin{aligned} V^*(\mathbf{x}) &\geqslant \sum_0^\infty (\delta^2)^t u^*(x_t, x_{t+1}) \\ &\geqslant \sum_0^\infty (\delta^2)^t \left[u(k_{2t}, k_{2t+1}) + \delta u(k_{2t+1}, k_{2t+2})\right] \\ &= \sum_0^\infty \delta^t u(k_t, k_{t+1}) = V(\mathbf{x}) = V^*(\mathbf{x}). \end{aligned}$$

Thus we must have equalities in all the above inequalities, so that $\{x_t\}$ is $\mathfrak{I}$-optimal from $\mathbf{x}$. This establishes that $h^*(\mathbf{x}) = h^2(\mathbf{x})$, and since $\mathbf{x} \geqslant 0$ was arbitrary, $h^* = h^2$.

THEOREM 3. *Suppose (Ω, u, δ) is a dynamic optimization model with policy function h. Let $k > 1$ be any odd integer. If h has a periodic orbit of period k, then*

$$\delta < (\sqrt{5} - 1)/2. \tag{7.1}$$

Proof. If h has a periodic orbit of period k where $k > 1$ is an odd integer, then by Sarkovskii's result (Proposition 1), h has a periodic orbit of period 6. This implies that h^2 has a periodic orbit of period 3.

By Proposition 7, there exists a dynamic optimization model, $\mathfrak{I}^* = (\Omega^*, u^*, \delta^*)$, with $\delta^* = \delta^2$, such that the policy function, h^*, of $\mathfrak{I}^*$ is given by h^2. By using Theorem 2, we can conclude that

$$\delta^* < [(\sqrt{5} - 1)/2]^2.$$

Since $\delta^* = \delta^2$, (7.1) follows immediately.

Remarks. (i) Sorger [22] conjectured that one can find for every $\delta \in (0, 1)$ and odd number $p \geqslant 3$ and a continuous function h from $[0, 1]$ to $[0, 1]$, such that h is the policy function of a dynamic optimization model (Ω, u, δ), and h has a periodic point of period p. Theorem 3 indicates that this conjecture is false for $1 > \delta \geqslant (\sqrt{5} - 1)/2$.

(ii) The restriction in (7.1) on the *discount factor* translates to the *discount rate* restriction

$$\rho > (\sqrt{5} - 1)/2 \approx 0.618,$$

which still involves substantial discounting.

(iii) There is no claim in our result that the discount factor restriction (7.1) is the "best possible." In fact, applying the theory of turbulence to this problem, I have shown elsewhere (see Mitra [12]) that the bound in (7.1) can be improved.

The Sarkovskii ordering and the above method of proof can be used to obtain discount factor restrictions that must hold in order that optimal programs be periodic with period $p = 2^q.\ k$ where $q \geqslant 0$ and $k > 1$ is an odd integer.

COROLLARY 2. *Suppose* $\mathfrak{I} = (\Omega, u, \delta)$ *is a dynamic optimization model with policy function h. Let* $k > 1$ *be an odd integer, q be a non-negative integer, and* $p = 2^q.k$. *If h has a periodic orbit with period p, then*

$$\delta < [(\sqrt{5} - 1)/2]^{1/2^q}. \tag{7.2}$$

Remark. Nishimura and Yano [17] have constructed a sequence of dynamic optimization models $\mathfrak{I}_n \equiv (\Omega_n, u_n, \delta_n)$, for $n = 0, 1, 2, ...$, such that $\mathfrak{I}_n$ exhibits a periodic point of period 3.2^n and $\delta_n \to 1$ as $n \to \infty$. This appears to be in accord with the discount rate restrictions we obtain in our result not being uniformly below 1. For $n = 0, 1, 2, 3$, they also noted that in obtaining each successive model exhibiting a period twice that in the previous model (6 compared to 3, 12 compared to 6, 24 compared to 12), the (minimum) discount factor needed was approximately "square rooted." This feature also appears to be in accord with the *formula* we obtain for the discount rate restrictions.

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